

ON LAZAREVIĆ AND CUSA TYPE INEQUALITIES FOR HYPERBOLIC FUNCTIONS WITH TWO PARAMETERS

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ABSTRACT. In this paper, by investigating the monotonicity of a function composed of $(\sinh x)/x$ and $\cosh x$ with two parameters in x on $(0, \infty)$, we prove several theorems related to inequalities for hyperbolic functions, which generalize known results and establish some new and sharp inequalities. As applications, some new and sharp inequalities for bivariate means are presented.

1. INTRODUCTION

Lazarević [1] (or see Mitrinović [2]) proved that for $x \neq 0$, the inequality

$$(1.1) \quad \left(\frac{\sinh x}{x} \right)^q > \cosh x$$

holds if and only if $q \geq 3$. This result has been generalized by Zhu in [3] as follows.

Theorem Zhu1. *Let $p > 1$ or $p \leq 8/15$, and $x \in (0, \infty)$. Then*

$$\left(\frac{\sinh x}{x} \right)^q > p + (1 - p) \cosh x$$

if and only if $q \geq 3(1 - p)$.

Yang gave another generalization and refinement in [4].

Theorem Yang. *Let $p, x > 0$. Then the following inequality*

$$(1.2) \quad \frac{\sinh x}{x} > (\cosh px)^{1/(3p^2)}$$

holds for all $x > 0$ if and only if $p \geq p_0 = 1/\sqrt{5}$, and the function $p \mapsto (\cosh px)^{1/(3p^2)}$ is decreasing on $(0, \infty)$. Inequality (1.2) is reversed if and only if $0 < p \leq 1/3$.

Another inequality related to Lazarević inequality is the so-called Cusa type one (see [5]), which states that

$$(1.3) \quad \frac{\sinh x}{x} < \frac{2 + \cosh x}{3}$$

holds for $x > 0$.

In [6], Zhu established a more general result which contains Lazarević and Cusa-type inequalities.

Theorem Zhu2. *Let $x > 0$. Then the following are considered.*

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(i) If $p \geq 4/5$, the double inequality

$$1 - \lambda + \lambda (\cosh x)^p < \left(\frac{\sinh x}{x} \right)^p < 1 - \eta + \eta (\cosh x)^p$$

holds if and only if $\eta \geq 1/3$ and $\lambda \leq 0$.

(ii) If $p < 0$, the inequality

$$\left(\frac{\sinh x}{x} \right)^p < 1 - \eta + \eta (\cosh x)^p$$

holds if and only if $\eta \leq 1/3$.

That is, let $\alpha > 0$, then the inequality

$$\left(\frac{x}{\sinh x} \right)^\alpha < 1 - \eta + \eta \left(\frac{1}{\cosh x} \right)^\alpha$$

holds if and only if $\eta \leq 1/3$.

Other inequalities for hyperbolic functions can be found in [7], [8], [9], [5], [10], [11], [12], [13], [14], [15], [4], [16], [17], [18], and references therein.

The aim of this paper is to establish more general than Zhu's inequalities for hyperbolic functions. In Section 2, we investigate the monotonicity of the function $H_{p,q}$ defined on $(0, \infty)$ by

$$(1.4) \quad H_{p,q}(x) = \frac{U_p\left(\frac{\sinh x}{x}\right)}{U_q(\cosh x)},$$

where $p, q \in \mathbb{R}$ and U_p is defined on $(1, \infty)$ by

$$(1.5) \quad U_p(t) = \frac{t^p - 1}{p} \text{ if } p \neq 0 \text{ and } U_0(t) = \ln t.$$

If we can prove that $H_{p,q}$ is increasing or decreasing on $(0, \infty)$ for certain p, q , then we will obtain $H_{p,q}(x) > (\text{or } <) H_{p,q}(0^+) = 1/3$, which may yield some new inequalities for hyperbolic functions. Our main purpose in the section is to find the relations between p with q such that $H_{p,q}$ has monotonicity property. Based on them, many new sharp inequalities for hyperbolic functions are derived in Section 3. In the last section, some new sharp inequalities for bivariate means are presented.

2. MONOTONICITY

We begin with the following simple assertion.

Lemma 1. *Let the function U_p defined on $(1, \infty)$ by (1.5). Then $p \mapsto U_p(t)$ is increasing on \mathbb{R} and $U_p(t) > 0$ for $t \in (1, \infty)$.*

Proof. For $p \neq 0$, differentiation yields

$$\frac{\partial U_p(t)}{\partial p} = -\frac{1}{p^2} (t^p - 1) + \frac{1}{p} t^p \ln t = -\frac{t^p}{p^2} (\ln t^{-p} - (t^{-p} - 1)) > 0,$$

where the last inequality holds due to $\ln x \leq (x - 1)$ for $x > 0$.

Employing the decreasing property, we get

$$U_p(t) > \lim_{p \rightarrow -\infty} U_p(t) = \lim_{p \rightarrow -\infty} \frac{t^p - 1}{p} = 0,$$

which proves the lemma. \square

For $x \in (0, \infty)$, we denote by

$$Sh_p(x) := U_p\left(\frac{\sinh x}{x}\right) \quad \text{and} \quad Ch_p(x) := U_p(\cosh x)$$

due to $(\sinh x)/x, \cosh x \in (1, \infty)$. Then we have

$$(2.1) \quad Sh_p(x) = \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} \quad \text{if } p \neq 0 \quad \text{and} \quad Sh_0(x) = \ln \frac{\sinh x}{x} \quad \text{if } p = 0,$$

$$(2.2) \quad Ch_p(x) = \frac{\cosh^p x - 1}{p} \quad \text{if } p \neq 0 \quad \text{and} \quad Ch_0(x) = \ln(\cosh x) \quad \text{if } p = 0.$$

And then, the function $x \mapsto H_{p,q}(x) = U_p\left(\frac{\sinh x}{x}\right)/U_q(\cosh x) = Sh_p(x)/Ch_q(x)$ can be expressed as

$$(2.3) \quad H_{p,q}(x) = \begin{cases} \frac{q}{p} \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{\cosh^q x - 1} & \text{if } pq \neq 0, \\ \frac{1}{p} \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{\ln(\cosh x)} & \text{if } p \neq 0, q = 0, \\ q \frac{\ln \frac{\sinh x}{x}}{\cosh^q x - 1} & \text{if } p = 0, q \neq 0, \\ \frac{\ln \frac{\sinh x}{x}}{\ln(\cosh x)} & \text{if } p = q = 0. \end{cases}$$

In order to investigate the monotonicity of the function $H_{p,q}$, we first recall the following lemmas.

Lemma 2 ([20], [21]). *Let $f, g : [a, b] \mapsto \mathbb{R}$ be two continuous functions which are differentiable on (a, b) . Further, let $g' \neq 0$ on (a, b) . If f'/g' is increasing (or decreasing) on (a, b) , then so are the functions*

$$x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)}.$$

Lemma 3 ([22]). *Let a_n and b_n ($n = 0, 1, 2, \dots$) be real numbers and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \dots$, and a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \dots$, then the function $A(t)/B(t)$ is strictly increasing (or decreasing) on $(0, R)$.*

Now we are in position to prove the monotonicity of $H_{p,q}$. Clearly, $H_{p,q}(x)$ can be written as

$$H_{p,q}(x) = \frac{Sh_p(x)}{Ch_q(x)} = \frac{Sh_p(x) - Sh_p(0^+)}{Ch_q(x) - Ch_q(0^+)}.$$

For $pq \neq 0$, differentiation yields

$$(2.4) \quad \frac{Sh'_p(x)}{Ch'_q(x)} = \frac{\cosh^{1-q} x}{x^2 \sinh x} \left(\frac{\sinh x}{x}\right)^{p-1} (x \cosh x - \sinh x) := f_1(x)$$

$$(2.5) \quad f'_1(x) = \frac{1}{x^2 \sinh^3 x \cosh^q x} \left(\frac{\sinh x}{x}\right)^p \times f_2(x),$$

where

$$(2.6) \quad f_2(x) = pA(x) - qB(x) + C(x),$$

in which

$$(2.7a) \quad A(x) = (\sinh x - x \cosh x)^2 \cosh x > 0,$$

$$(2.7b) \quad B(x) = x(x \cosh x - \sinh x) \sinh^2 x > 0$$

$$(2.7c) \quad C(x) = -2x^2 \cosh x + x \sinh x + \cosh x \sinh^2 x > 0,$$

here $C(x) > 0$ due to

$$C(x) = x^2 (\cosh x) \left(\frac{\sinh^2 x}{x^2} + \frac{\tanh x}{x} - 2 \right) > 0$$

by Wilker type inequality (see [19]). It is easy to verify that (2.4), (2.5) and (2.6) are true for $pq = 0$.

Expanding in power series yields

$$\begin{aligned} A(x) &= \frac{1}{4}x^2 \cosh 3x + \frac{3}{4}x^2 \cosh x - \frac{1}{2}x \sinh 3x - \frac{1}{2}x \sinh x + \frac{1}{4} \cosh 3x - \frac{1}{4} \cosh x \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{2n+2} + \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n+2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n-1)!} x^{2n} \\ &\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} x^{2n} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{2n} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \\ (2.8) \quad &= \sum_{n=3}^{\infty} \frac{((4n^2 - 14n + 9)9^{n-1} + 12n^2 - 10n - 1)}{4(2n)!} x^{2n} := \sum_{n=3}^{\infty} \frac{a_n}{4(2n)!} x^{2n}, \end{aligned}$$

$$\begin{aligned} B(x) &= \frac{1}{4}x^2 \cosh 3x - \frac{1}{4}x^2 \cosh x - \frac{1}{4}x \sinh 3x + \frac{3}{4}x \sinh x \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{2n+2} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n+2} \\ &\quad - \frac{1}{4} \sum_{n=1}^{\infty} \frac{3^{2n-1}}{(2n-1)!} x^{2n} + \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} x^{2n} \\ (2.9) \quad &= \sum_{n=3}^{\infty} \frac{(4n(n-2)9^{n-1} - 4n(n-2))}{4(2n)!} x^{2n} := \sum_{n=3}^{\infty} \frac{b_n}{4(2n)!} x^{2n}, \end{aligned}$$

$$\begin{aligned} C(x) &= -2x^2 \cosh x + x \sinh x + \frac{1}{4} \cosh 3x - \frac{1}{4} \cosh x \\ &= -2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n+2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} x^{2n} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{2n} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \\ (2.10) \quad &= \sum_{n=3}^{\infty} \frac{(9^n - 32n^2 + 24n - 1)}{4(2n)!} x^{2n} := \sum_{n=3}^{\infty} \frac{c_n}{4(2n)!} x^{2n}. \end{aligned}$$

We see clearly that, by Lemma 2, if we can prove $f_2(x) \geq (\leq) 0$ for all $x \in (0, \infty)$ then $H_{p,q}$ defined by (2.3) is increasing (decreasing) on $(0, \infty)$. To this end, we need to prove the following important statement.

Lemma 4. Let f_3 be defined on $(0, \infty)$ by

$$(2.11) \quad f_3(x) = \frac{pA(x) - qB(x)}{C(x)} + 1.$$

where $A(x)$, $B(x)$ and $C(x)$ are defined by (2.7a), (2.7b) and (2.7c), respectively. Then

(i) f_3 is strictly increasing on $(0, \infty)$ if $(p, q) \in \mathbb{I}_1$, where

$$(2.12) \quad \mathbb{I}_1 = \{q = 0, p > 0\} \cup \left\{q > 0, \frac{p}{q} \geq \frac{23}{17}\right\} \cup \left\{q < 0, \frac{p}{q} \leq 1\right\},$$

and we have

$$\frac{5}{8}p - \frac{15}{8}q + 1 < f_3(x) < \infty;$$

(ii) f_3 is strictly decreasing on $(0, \infty)$ if $(p, q) \in \mathbb{I}_2$, where

$$(2.13) \quad \mathbb{I}_2 = \{q = 0, p < 0\} \cup \left\{q > 0, \frac{p}{q} \leq 1\right\} \cup \left\{q < 0, \frac{p}{q} \geq \frac{23}{17}\right\},$$

and we have

$$-\infty < f_3(x) < \frac{5}{8}p - \frac{15}{8}q + 1.$$

Proof. Using (2.8), (2.9) and (2.10) gives

$$f_3(x) - 1 = \frac{pA(x) - qB(x)}{C(x)} = \frac{\sum_{n=3}^{\infty} \frac{(pa_n - qb_n)}{4(2n)!} x^{2n}}{\sum_{n=3}^{\infty} \frac{c_n}{4(2n)!} x^{2n}},$$

where

$$(2.14) \quad a_n = ((4n^2 - 14n + 9)9^{n-1} + 12n^2 - 10n - 1),$$

$$(2.15) \quad b_n = (4n(n-2)9^{n-1} - 4n(n-2)),$$

$$(2.16) \quad c_n = (9^n - 32n^2 + 24n - 1).$$

In order to observe the monotonicity of f_3 , we need to investigate the monotonicity of series

$$\frac{(pa_n - qb_n) / (4(2n)!)}{c_n / (4(2n)!)} = \frac{pa_n - qb_n}{c_n}.$$

We have

$$\begin{aligned} & \frac{pa_{n+1} - qb_{n+1}}{c_{n+1}} - \frac{pa_n - qb_n}{c_n} \\ &= \frac{p(a_{n+1}c_n - a_nc_{n+1}) - q(b_{n+1}c_n - b_nc_{n+1})}{c_nc_{n+1}} \\ &= \frac{pv_n - qu_n}{c_nc_{n+1}} = \begin{cases} p \frac{v_n}{c_nc_{n+1}} & \text{if } q = 0, \\ \frac{v_n}{c_nc_{n+1}} q \left(\frac{p}{q} - \frac{u_n}{v_n} \right) & \text{if } q \neq 0, \end{cases} \end{aligned}$$

where

$$\begin{aligned} u_n &= b_{n+1}c_n - b_nc_{n+1} \\ &= 2 \times 9^{n-1} [(36n - 18)9^n - (512n^4 - 384n^3 - 560n^2 + 792n - 36)] \\ &\quad + 4(42n + 40n^2 - 1), \end{aligned}$$

$$\begin{aligned}
v_n &= a_{n+1}c_n - a_nc_{n+1} \\
&= 2 \times 9^{n-1} [(36n - 45)9^{2n} - (512n^4 - 1152n^3 + 1072n^2)] \\
&\quad + 2 \times [2(28n + 5)9^n - (16n^2 + 60n + 5)] \\
&: = 2 \times 9^{n-1}v'_n + 2v''_n.
\end{aligned}$$

Now we distinguish three cases to discuss the monotonicity of f_3 .

(i) When $q = 0$, we have $c_n, v_n > 0$ for $n \geq 3$. Indeed, we use binomial expansion to get

$$\begin{aligned}
c_n &= 9^n - 32n^2 + 24n - 1 = (1 + 8)^n - 32n^2 + 24n - 1 \\
&> 1 + 8n + \frac{n(n-1)}{2}8^2 - 32n^2 + 24n - 1 = 0.
\end{aligned}$$

Application of binomial expansion again we have

$$\begin{aligned}
v'_n &= (36n - 45)(1 + 8)^n - (512n^4 - 1152n^3 + 1072n^2) \\
&> (36n - 45) \left(1 + 8n + \frac{n(n-1)}{2}8^2 + \frac{n(n-1)(n-2)}{6}8^3 \right) - (512n^4 - 1152n^3 + 1072n^2) \\
&= 2560n^4 - 10752n^3 + 14288n^2 - 6564n - 45 \\
&= 2560(n-3)^4 + 19968(n-3)^3 + 55760(n-3)^2 + 65340(n-3) + 25911 > 0,
\end{aligned}$$

$$\begin{aligned}
v''_n &= 2(28n + 5)(1 + 8)^n - (16n^2 + 60n + 5) \\
&> 2(28n + 5)(1 + 8n) - (16n^2 + 60n + 5) \\
&= 432n^2 + 76n + 5 > 0
\end{aligned}$$

which show that $v_n = v'_n + v''_n > 0$ for $n \geq 3$. Thus, $(pa_n - qb_n)/c_n$ is increasing if $p \geq 0$ and decreasing if $p < 0$, and by Lemma 3 so is $f_3 - 1$ on $(0, \infty)$. Hence, we have

$$\begin{aligned}
\frac{5}{8}p + 1 &= \lim_{x \rightarrow 0^+} f_3(x) < f_3(x) < \lim_{x \rightarrow \infty} f_3(x) = \infty \text{ if } p > 0, \\
f_3(x) &= 1 \text{ if } p = 0, \\
-\infty &= \lim_{x \rightarrow \infty} f_3(x) < f_3(x) < \lim_{x \rightarrow 0^+} f_3(x) = \frac{5}{8}p + 1 \text{ if } p < 0.
\end{aligned}$$

(ii) When $q \neq 0$, we claim that u_n/v_n is decreasing for $n \geq 3$. Since $v_n > 0$ for $n \geq 3$, it suffices to show that $u_nv_{n+1} - u_{n+1}v_n > 0$. Factoring and arranging give us to

$$\begin{aligned}
\frac{u_nv_{n+1} - u_{n+1}v_n}{c_{n+1}} &= a_{n+2}(c_nb_{n+1} - b_nc_{n+1}) + b_{n+2}(a_nc_{n+1} - c_na_{n+1}) \\
+c_{n+2}(b_na_{n+1} - a_nb_{n+1}) &= \frac{16}{3}w_n,
\end{aligned}$$

where

$$\begin{aligned}
w_n &= 9^{3n+2} - (1024n^4 - 2560n^3 + 2752n^2 + 243)9^{2n} \\
(2.17) \quad &+ (1024n^4 + 2560n^3 + 2752n^2 + 243)9^n - 81.
\end{aligned}$$

As shown previously, $c_{n+1} > 0$ for $n \geq 3$, and we only need to prove $w_n > 0$ for $n \geq 3$. Since the sum of the third and fourth terms in (2.17) is obviously positive,

and it suffices to show that the sum of the first and second is also positive. Using binomial expansion again, we have

$$\begin{aligned}
9^{-2n}w_n &> 9^{n+2} - (1024n^4 - 2560n^3 + 2752n^2 + 243), \\
&> 1 + 8(n+2) + \frac{(n+2)(n+1)}{2}8^2 + \frac{(n+2)(n+1)n}{6}8^3 + \frac{(n+2)(n+1)n(n-1)}{24}8^4 \\
&\quad + \frac{(n+2)(n+1)n(n-1)(n-2)}{120}8^5 - (1024n^4 - 2560n^3 + 2752n^2 + 243) \\
&= \frac{2}{15} (2048n^5 - 6400n^4 + 12160n^3 - 19760n^2 + 7692n - 1215) \\
&= 2048(n-3)^5 + 24320(n-3)^4 + 119680(n-3)^3 \\
&\quad + 297040(n-3)^2 + 355692(n-3) + 151605 \\
&> 0,
\end{aligned}$$

which proves the decreasing property of u_n/v_n for $n \geq 3$. It follows that

$$1 = \lim_{n \rightarrow \infty} \frac{u_n}{v_n} < \frac{u_n}{v_n} \leq \frac{u_3}{v_3} = \frac{23}{17},$$

and then, we conclude that

$$\frac{pa_{n+1}-qb_{n+1}}{c_{n+1}} - \frac{pa_n-qb_n}{c_n} = \frac{v_n}{c_n c_{n+1}} q \left(\frac{p}{q} - \frac{u_n}{v_n} \right) \begin{cases} > 0 & \text{if } q > 0, \frac{p}{q} \geq \frac{23}{17}, \\ < 0 & \text{if } q < 0, \frac{p}{q} \geq \frac{23}{17}, \\ < 0 & \text{if } q > 0, \frac{p}{q} \leq 1, \\ > 0 & \text{if } q < 0, \frac{p}{q} \leq 1, \end{cases}$$

which by Lemma 3 yield the desired monotonicity results. And, easy calculations gives

$$\begin{aligned}
\lim_{x \rightarrow 0^+} f_3(x) - 1 &= \frac{5}{8}p - \frac{15}{8}q, \\
\lim_{x \rightarrow \infty} f_3(x) - 1 &= \begin{cases} \infty & \text{if } q > 0, \frac{p}{q} \geq \frac{23}{17} \text{ or } q < 0, \frac{p}{q} \leq 1, \\ -\infty & \text{if } q < 0, \frac{p}{q} \geq \frac{23}{17} \text{ or } q > 0, \frac{p}{q} \leq 1. \end{cases}
\end{aligned}$$

Thus the proof is completed \square

From the lemma above, we easily get the monotonicity of $H_{p,q}$.

Proposition 1. *Let $H_{p,q}$ be defined on $(0, \infty)$ by (2.3). Then*

(i) $H_{p,q}$ is increasing on $(0, \infty)$ if

$$(p, q) \in (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 \geq 0 \right\},$$

where \mathbb{I}_1 is defined by (2.12);

(ii) $H_{p,q}$ is decreasing on $(0, \infty)$ if

$$(p, q) \in \mathbb{I}_2 \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 \leq 0 \right\},$$

where \mathbb{I}_2 is defined by (2.13).

Proof. As mentioned previously, to prove the monotonicity of $H_{p,q}$, it suffices to deal with the sings of $f_2(x)$ on $(0, \infty)$. It is clear that

$$\frac{f_2(x)}{C(x)} = f_3(x),$$

where $f_3(x)$ is defined by (2.11). Then, $\operatorname{sgn} f_2(x) = \operatorname{sgn} f_3(x)$ due to $C(x) > 0$ for $x \in (0, \infty)$.

(i) When $(p, q) \in \mathbb{I}_1 \cup \{(0, 0)\}$, it is obtained from Lemma 4 that $f_2(x) > 0$ for $x \in (0, \infty)$ provided $\inf_{x>0} f_3(x) = 5p/8 - 15q/8 + 1 \geq 0$. Utilizing the relation (2.5) and Lemma 2 we get the conclusion that $H_{p,q}$ is increasing on $(0, \infty)$ for $(p, q) \in (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \{5p/8 - 15q/8 + 1 \geq 0\}$.

(ii) When $(p, q) \in \mathbb{I}_2$, $f_2(x) < 0$ for $x \in (0, \infty)$ so long as $\sup_{x>0} f_3(x) = 5p/8 - 15q/8 + 1 \leq 0$. Then, $H_{p,q}$ is decreasing on $(0, \infty)$ for $(p, q) \in \mathbb{I}_2 \cap \{5p/8 - 15q/8 + 1 \leq 0\}$.

Thus we complete the proof. \square

It is easy to check that $(p, q) \in (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \{\frac{5}{8}p - \frac{15}{8}q + 1 \geq 0\}$ is equivalent to

$$p \geq \begin{cases} 3q - \frac{8}{5} & \text{if } q \in [\frac{34}{35}, \infty) \\ \frac{23}{17}q & \text{if } q \in [0, \frac{34}{35}), \\ q & \text{if } q \in (-\infty, 0), \end{cases}$$

while $(p, q) \in \mathbb{I}_2 \cap \{\frac{5}{8}p - \frac{15}{8}q + 1 \leq 0\}$ is equivalent to

$$p \leq \begin{cases} q & \text{if } q \in [\frac{4}{5}, \infty), \\ 3q - \frac{8}{5} & \text{if } q \in (-\infty, \frac{4}{5}). \end{cases}$$

Then Proposition 1 can be restated as follows.

Proposition 2. *Let $H_{p,q}$ be defined on $(0, \infty)$ by (2.3). Then*

(i) *when $q \in [34/35, \infty)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $p \geq 3q - 8/5$ and decreasing for $p \leq q$;*

(ii) *when $q \in [4/5, 34/35)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $p \geq 23q/17$ and decreasing for $p \leq q$;*

(iii) *when $q \in (0, 4/5)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $p \geq 23q/17$ and decreasing for $p \leq 3q - 8/5$;*

(iv) *when $q \in (-\infty, 0]$, $H_{p,q}$ is increasing on $(0, \infty)$ for $p \geq q$ and decreasing for $p \leq 3q - 8/5$.*

On the other hand, $(p, q) \in (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \{\frac{5}{8}p - \frac{15}{8}q + 1 \geq 0\}$ is equivalent to

$$q \leq \begin{cases} \frac{p}{3} + \frac{8}{15} & \text{if } p \in [\frac{46}{35}, \infty) \\ \frac{17}{23}p & \text{if } p \in [0, \frac{46}{35}), \\ p & \text{if } p \in (-\infty, 0), \end{cases}$$

while $(p, q) \in \mathbb{I}_2 \cap \{\frac{5}{8}p - \frac{15}{8}q + 1 \leq 0\}$ is equivalent to

$$q \geq \begin{cases} p & \text{if } p \in [\frac{4}{5}, \infty), \\ \frac{p}{3} + \frac{8}{15} & \text{if } p \in (-\infty, \frac{4}{5}). \end{cases}$$

Then Proposition 1 also can be restated in another equivalent assertion.

Proposition 3. *Let $H_{p,q}$ be defined on $(0, \infty)$ by (2.3). Then*

(i) *when $p \in [46/35, \infty)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $q \leq p/3 + 8/15$ and decreasing for $q \geq p$;*

- (ii) when $p \in [4/5, 46/35)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $q \leq 17p/23$ and decreasing for $q \geq p$;
- (iii) when $p \in (0, 4/5)$, $H_{p,q}$ is increasing on $(0, \infty)$ for $q \leq 17p/23$ and decreasing for $q \geq p/3 + 8/15$;
- (iv) when $p \in (-\infty, 0]$, $H_{p,q}$ is increasing on $(0, \infty)$ for $q \leq p$ and decreasing for $q \geq p/3 + 8/15$.

Put $p = kq$, then by Proposition 1 in combination with its proof, we have

Corollary 1. *Let $H_{p,q}$ be defined on $(0, \infty)$ by (2.3). Then*

- (i) when $k \in (3, \infty)$, $H_{kq,q}$ is increasing for $q \geq 0$ and decreasing for $q \leq 8/(5(3-k))$;
- (ii) when $k = 3$, $H_{kq,q}$ is increasing for $q \in \mathbb{R}$;
- (iii) when $k \in [23/17, 3)$, $H_{kq,q}$ is increasing for $0 \leq q \leq 8/(5(3-k))$;
- (iv) when $k \in (1, 23/17)$, $H_{kq,q}$ is increasing for $q = 0$;
- (v) when $k \in (-\infty, 1]$, $H_{kq,q}$ is increasing for $q \leq 0$ and decreasing for $q \geq 8/(5(3-k))$.

If $5p/8 - 15q/8 + 1 = 0$, that is, $p = 3q - 8/5$ or $q = p/3 + 8/15$, then we easily check that

$$\begin{aligned} & (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 = 0 \right\} \\ &= \left\{ q \geq \frac{34}{35}, p = 3q - \frac{8}{5} \right\} = \left\{ p \geq \frac{46}{35}, q = \frac{p}{3} + \frac{8}{15} \right\}, \end{aligned}$$

$$\begin{aligned} & \mathbb{I}_2 \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 = 0 \right\} \\ &= \left\{ q \leq \frac{4}{5}, p = 3q - \frac{8}{5} \right\} = \left\{ p \leq \frac{4}{5}, q = \frac{p}{3} + \frac{8}{15} \right\}, \end{aligned}$$

and then by Proposition 1 we get

Corollary 2. *Let $H_{p,q}$ be defined on $(0, \infty)$ by (2.3). Then $H_{3q-8/5,q}$ is increasing if $q \geq 34/35$ and decreasing if $q \leq 4/5$. In other words, $H_{p,p/3+8/15}$ is increasing if $p \geq 46/35$ and decreasing if $p \leq 4/5$.*

3. RESULTS

In this section, we will give some new inequalities involving hyperbolic functions by using monotonicity theorems given in previous section.

Note that

$$\begin{aligned} & (\mathbb{I}_1 \cup \{(0, 0)\}) \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 \geq 0 \right\} \\ &= \{0 \leq q \leq \min(17p/23, p/3 + 8/15)\} \cup \{q \leq \min(0, p, p/3 + 8/15)\}, \end{aligned}$$

$$\begin{aligned} & \mathbb{I}_2 \cap \left\{ \frac{5}{8}p - \frac{15}{8}q + 1 \leq 0 \right\} \\ &= \{\max(17p/23, p/3 + 8/15) \leq q < 0\} \cup \{q \geq \max(0, p, p/3 + 8/15)\} \end{aligned}$$

and that $H_{p,q}(0^+) < (>) H_{p,q}(0^+) = 1/3$ is equivalent to $Sh_p(x) < (>) (1/3)Ch_q(x)$ for $x \in (0, \infty)$. By Proposition 1, we obtain the following theorem immediately.

Theorem 1. (i) If $0 \leq q \leq \min(17p/23, p/3 + 8/15)$ or $q \leq \min(0, p, p/3 + 8/15)$, then the inequalities

$$(3.1) \quad \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} > \frac{1}{3} \frac{\cosh^q x - 1}{q} \quad \text{if } pq \neq 0,$$

$$(3.2) \quad \ln \frac{\sinh x}{x} > \frac{1}{3} \frac{\cosh^q x - 1}{q} \quad \text{if } p = 0, q \neq 0,$$

$$(3.3) \quad \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} > \frac{1}{3} \ln \cosh x \quad \text{if } p \neq 0, q = 0,$$

$$(3.4) \quad \ln \frac{\sinh x}{x} > \frac{1}{3} \ln \cosh x \quad \text{if } p = q = 0,$$

hold for $x \in (0, \infty)$, where $1/3$ is the best constant.

(ii) If $\max(17p/23, p/3 + 8/15) \leq q < 0$ or $q \geq \max(0, p, p/3 + 8/15)$, then (3.1), (3.2) and (3.3) are reversed.

For clarity of expressions, in what follows we will directly write $Sh_p(x)$, $Ch_q(x)$, $H_{p,q}(x)$ etc. by their general formulas, and if $pq = 0$, then we regard them as limits at $p = 0$ or $q = 0$, unless otherwise specified. Now we are ready to establish sharp inequalities for hyperbolic by Propositions 2 and 3, Corollaries 1 and 2. To this end, we need a lemma.

Lemma 5. Let $D_{p,q}$ be defined on $(0, \infty)$ by

$$(3.5) \quad D_{p,q}(x) = Sh_p(x) - \frac{1}{3} Ch_q(x) = \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} - \frac{(\cosh x)^q - 1}{3q}.$$

(i) We have

$$(3.6) \quad \lim_{x \rightarrow 0^+} \frac{D_{p,q}(x)}{x^4} = \frac{1}{72} \left(p - 3q + \frac{8}{5} \right),$$

$$(3.7) \quad \lim_{x \rightarrow 0^+} \frac{D_{3q-8/5,q}(x)}{x^6} = \frac{1}{270} \left(q - \frac{34}{35} \right).$$

(ii) For $p, q \geq 0$, we have

$$(3.8) \quad \lim_{x \rightarrow \infty} e^{-qx} D_{p,q}(x) = \begin{cases} \infty & \text{if } p > q \geq 0, \\ \infty & \text{if } p \geq q = 0, \\ -\frac{2^{-q}}{3q} & \text{if } q \geq p > 0, \\ -\frac{2^{-q}}{3q} & \text{if } q > p = 0; \end{cases}$$

for other cases, we have

$$(3.9) \quad \lim_{x \rightarrow \infty} D_{p,q}(x) = \begin{cases} \infty & \text{if } p \geq 0, q < 0, \\ -\infty & \text{if } p < 0, q \geq 0, \\ \frac{1}{3q} - \frac{1}{p} & \text{if } p < 0, q < 0. \end{cases}$$

Proof. (i) For $pq \neq 0$, expanding in power series yields

$$(3.10) \quad \begin{aligned} D_{p,q}(x) &= \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} - \frac{(\cosh x)^q - 1}{3q} \\ &= \frac{5p-15q+8}{360} x^4 + \frac{35p^2-42p-315q^2+630q-320}{45360} x^6 + o(x^8), \end{aligned}$$

which leads to (3.6). It is easy to check that it holds for $p = 0$ or $q = 0$.

If $p = 3q - 8/5$, then we have

$$D_{p,q}(x) = \frac{35q - 34}{9450}x^6 + o(x^8),$$

which implies (3.7).

(ii) For $p, q > 0$, we have

$$\begin{aligned} \frac{D_{p,q}(x)}{e^{qx}} &= \frac{e^{-qx}}{p} \left(\frac{e^x - e^{-x}}{2x} \right)^p - \frac{e^{-qx}}{p} - \frac{e^{-qx}}{3q} \left(\frac{e^x + e^{-x}}{2} \right)^q + \frac{e^{-qx}}{3q} \\ &= \frac{1}{p} \frac{e^{(p-q)x}}{x^p} \left(\frac{1 - e^{-2x}}{2} \right)^p - \frac{1}{3q} \left(\frac{1 + e^{-2x}}{2} \right)^q - \left(\frac{1}{p} - \frac{1}{3q} \right) e^{-qx} \\ &\rightarrow \begin{cases} \infty & \text{if } p > q > 0, \\ -\frac{2^{-q}}{3q} & \text{if } q \geq p > 0, \end{cases} \quad \text{as } x \rightarrow \infty; \end{aligned}$$

for $p = 0, q > 0$, we have

$$\begin{aligned} e^{-qx} D_{0,q}(x) &= e^{-qx} \ln \frac{e^x - e^{-x}}{2x} - \frac{e^{-qx}}{3q} \left(\frac{e^x + e^{-x}}{2} \right)^q + \frac{e^{-qx}}{3q} \\ &= x e^{-qx} + e^{-qx} \ln \frac{1 - e^{-2x}}{2} - e^{-qx} \ln x - \frac{1}{3q} \left(\frac{1 + e^{-2x}}{2} \right)^q + \frac{e^{-qx}}{3q} \\ &\rightarrow -\frac{2^{-q}}{3q}, \quad \text{as } x \rightarrow \infty; \end{aligned}$$

for $p = 0, q = 0$, we have

$$\begin{aligned} D_{0,0}(x) &= \ln \frac{e^x - e^{-x}}{2x} - \frac{1}{3} \ln \frac{e^x + e^{-x}}{2} \\ &= x + \ln \frac{1 - e^{-2x}}{2} - \ln x - \frac{x}{3} - \frac{1}{3} \frac{1 + e^{-2x}}{2} \\ &= x \left(\frac{2}{3} - \frac{\ln x}{x} \right) + \ln \frac{1 - e^{-2x}}{2} - \frac{1}{3} \frac{1 + e^{-2x}}{2} \\ &\rightarrow \infty, \quad \text{as } x \rightarrow \infty; \end{aligned}$$

for $p > 0, q = 0$, utilizing the increasing property of $Sh_p(x) = U_p((\sinh x)/x)$, we get $Sh_p(x) > Sh_0(x)$, and then, $\lim_{x \rightarrow \infty} D_{p,0}(x) = \lim_{x \rightarrow \infty} D_{0,0}(x) = \infty$, which gives $\lim_{x \rightarrow \infty} D_{p,0}(x) = \infty$.

To sum up, relation (3.8) hold.

While (3.9) follows from the fact that for $t > 1$

$$U_p(\infty) = \lim_{t \rightarrow \infty} \frac{t^p - 1}{p} = \infty \text{ if } p \geq 0, U_p(\infty) = \lim_{t \rightarrow \infty} \frac{t^p - 1}{p} = -\frac{1}{p} \text{ if } p < 0,$$

which proves the lemma. \square

Utilizing Proposition 2 and lemma above we have the following theorem.

Theorem 2. *Let $x \in (0, \infty)$. Then*

(i) when $q \in [34/35, \infty)$, the double inequality

$$(MI2) \quad \frac{\left(\frac{\sinh x}{x} \right)^{p_2} - 1}{p_2} < \frac{\cosh^q x - 1}{3q} < \frac{\left(\frac{\sinh x}{x} \right)^{p_1} - 1}{p_1}$$

holds if and only if $p_1 \geq 3q - 8/5$ and $p_2 \leq q$;

- (ii) when $q \in [4/5, 34/35)$, the double inequality (MI2) holds for $p_1 \geq 23q/17$ and if and only if $p_2 \leq q$;
- (iii) when $q \in (0, 4/5)$, the double inequality (MI2) holds for $p_1 \geq 23q/17$ and if and only if $p_2 \leq 3q - 8/5$;
- (iv) when $q \in (-\infty, 0]$, the double inequality (MI2) holds if and only if $p_1 \geq q$ and $p_2 \leq 3q - 8/5$.

Proof. The sufficiencies in the cases of (i)–(iv) are due to Proposition 2. Now we show the necessities in certain cases.

(i) When $q \in [34/35, \infty)$, the condition $p_1 \geq 3q - 8/5$ is necessary for the second inequality in (MI2) to hold. If the second in (MI2) holds, then we have $\lim_{x \rightarrow 0^+} x^{-4} D_{p_1, q}(x) \geq 0$, which, by (3.6), yields $p_1 \geq 3q - 8/5$. We claim that the condition $p_2 \leq q$ is also necessary for the first inequality in (MI2) to be true. If there is a $p_2 > q \in [34/35, \infty)$ such that the first inequality in (MI2) holds, then by (3.9) there must be $\lim_{x \rightarrow \infty} e^{-qx} D_{p_2, q}(x) = \infty$, which yields a contradiction. Hence, the condition $p_2 > q$ is also necessary.

(ii) When $q \in [4/5, 34/35)$, similar to part two of proof (i), the condition $p_2 \leq q$ is necessary for the first inequality in (MI2) to be valid.

(iii) When $q \in (0, 4/5)$, in the same way as part one of proof (i), the condition $p_2 \leq 3q - 8/5$ is necessary for the first inequality in (MI2) to hold.

(iv) When $q \in (-\infty, 0)$, analogous to the case of $q \in [34/35, \infty)$, we can prove the conditions $p_1 \geq q$ and $p_2 \leq 3q - 8/5$ are necessary.

This completes the proof. \square

Remark 1. Taking $k = 1$ in Theorem 2, we get a equivalent result of Theorem Zhu1.

Similarly, by Proposition 3 and Lemma 5 we can prove the following statement.

Theorem 3. Let $x \in (0, \infty)$. Then

(i) when $p \in [46/35, \infty)$, the double inequality

$$(MI3) \quad \frac{\cosh^{q_1} x - 1}{3q_1} < \frac{\left(\frac{\sinh x}{x}\right)^p - 1}{p} < \frac{\cosh^{q_2} x - 1}{3q_2}$$

holds if and only if $q_1 \leq p/3 + 8/15$ and $q_2 \geq p$;

(ii) when $p \in [4/5, 46/35)$, the double inequality (MI3) holds for $q_1 \leq 17p/23$ and if and only if $q_2 \geq p$;

(iii) when $p \in (0, 4/5)$, the double inequality (MI3) holds for $q_1 \leq 17p/23$ and if and only if $q_2 \geq p/3 + 8/15$;

(iv) when $p \in (-\infty, 0]$, the double inequality (MI3) holds if and only if $q_1 \leq p$ and $q_2 \geq p/3 + 8/15$.

Remark 2. (i) For $x \in (0, \infty)$, $Sh_p(x) < (>) (1/3)Ch_q(x)$ is equivalent to $(\sinh x)/x > (<) M(\cosh x; p, q)$ for certain $(p, q) \in \Omega_{p, q}$, where

$$(3.11) \quad M(t; p, q) = \begin{cases} \left(1 - \frac{p}{3q} + \frac{p}{3q}t^q\right)^{1/p} & \text{if } pq \neq 0, (p, q) \in \Omega_{p, q}, \\ \exp \frac{t^q - 1}{3q} & \text{if } p = 0, q \neq 0, \\ \left(\frac{p}{3} \ln t + 1\right)^{1/p} & \text{if } p > 0, q = 0, \\ t^{1/3} & \text{if } p = q = 0, \end{cases}$$

here $t = \cosh x \in (1, \infty)$. It is easy to verify that for $t \in (1, \infty)$, the largest set of (p, q) such that $M(t; p, q)$ exists in real number field is

$$(3.12) \quad \Omega_{p,q} = \{(p, q) : p \geq 0 \text{ or } 3q \leq p \leq 0\}.$$

(ii) We suggest that M is decreasing in p and increasing in q if $(p, q) \in \Omega_{p,q}$.

In fact, for $(p, q) \in \Omega_{p,q}$ with $pq \neq 0$, logarithmic differentiation yields

$$\begin{aligned} \frac{\partial \ln M}{\partial p} &= \frac{1}{p^2} \left(-\ln \left(1 - \frac{p}{3q} + \frac{p}{3q} t^q \right) - \frac{p(1-t^q)}{(pt^q + 3q - p)} \right) := \frac{M_1(t; p, q)}{p^2}, \\ \frac{\partial M_1}{\partial p} &= -\frac{p(1-t^q)^2}{(pt^q + 3q - p)^2}, \end{aligned}$$

which implies that M_1 is decreasing in p on $(0, \infty)$ and increasing on $(-\infty, 0)$. Hence we have $M_1(t; p, q) < M_1(t; 0, q) = 0$, which means that M is decreasing in p .

It is easy to check that the monotonicity result of M in p is also true for $pq = 0$.

Similarly, we have

$$\frac{\partial \ln M}{\partial q} = -\frac{t^q (\ln t^{-q} - t^{-q} + 1)}{3q^2 \left(\frac{p}{3q} t^q + 1 - \frac{p}{3q} \right)} > 0,$$

where the inequality holds due to $\ln x \leq x - 1$ for $x > 0$ and $(p/(3q))t^q + 1 - (p/(3q)) > 0$ for $(t, p, q) \in (1, \infty) \times \Omega_{p,q}$, which proves the monotonicity of M with respect to q .

Remark 3. By Remark above, if we add the condition that " $(p, q) \in \Omega_{p,q}$ " in Theorems 2 and 3, and replace (MI2), (MI3) with

$$(MI2^*) \quad \left(1 - \frac{p_1}{3q} + \frac{p_1}{3q} \cosh^q x \right)^{1/p_1} < \frac{\sinh x}{x} < \left(1 - \frac{p_2}{3q} + \frac{p_2}{3q} \cosh^q x \right)^{1/p_2},$$

$$(MI3^*) \quad \left(1 - \frac{p}{3q_1} + \frac{p}{3q_1} \cosh^{q_1} x \right)^{1/p} < \frac{\sinh x}{x} < \left(1 - \frac{p}{3q_2} + \frac{p}{3q_2} \cosh^{q_2} x \right)^{1/p},$$

respectively, then the two theorems are still true.

Taking $q = 1$ in Theorem 2 and notice that $(p, q) \in \Omega_{p,q}$, we get

Corollary 3. The double inequality

$$(3.13) \quad \left(1 - \frac{p_1}{3} + \frac{p_1}{3} \cosh x \right)^{1/p_1} < \frac{\sinh x}{x} < \left(1 - \frac{p_2}{3} + \frac{p_2}{3} \cosh x \right)^{1/p_2}$$

holds if and only if $p_1 \geq 7/5$ and $0 \leq p_2 \leq 1$.

Remark 4. Letting $p_1 = 7/5, 3/2, 2, 3$ and using the decreasing property of $M(\cosh x; p, q)$ with respect to p , we can obtain the following chain of inequalities from (3.13):

$$\begin{aligned} \cosh^{1/3} x &< \left(\frac{1}{3} + \frac{2}{3} \cosh x \right)^{1/2} < \left(\frac{1}{2} + \frac{1}{2} \cosh x \right)^{2/3} \\ &< \left(\frac{8}{15} + \frac{7}{15} \cosh x \right)^{5/7} < \frac{\sinh x}{x} < \frac{2}{3} + \frac{1}{3} \cosh x. \end{aligned}$$

Clearly, this chain of inequalities is superior to Che and Sándor's given in [17, (3.23)].

Taking $p = 0, 1$ in Theorem 3 and notice that $(p, q) \in \Omega_{p,q}$, we get

Corollary 4. (i) *The double inequality*

$$(3.14) \quad \exp \frac{\cosh^{q_1} x - 1}{3q_1} < \frac{\sinh x}{x} < \exp \frac{\cosh^{q_2} x - 1}{3q_2}$$

holds if and only if $q_1 \leq 0$ and $q_2 \geq 8/15$.

(ii) *The double inequality*

$$(3.15) \quad 1 - \frac{1}{3q_1} + \frac{1}{3q_1} \cosh^{q_1} x < \frac{\sinh x}{x} < 1 - \frac{1}{3q_2} + \frac{1}{3q_2} \cosh^{q_2} x$$

holds for $q_1 \leq 17/23 \approx 0.73913$ and if and only if $q_2 \geq 1$.

Remark 5. Letting $q_1 = 17/23, 2/3, 1/2, 1/3, 1/6, 0$ and using the increasing property of $M(\cosh x; p, q)$ in q , we get the following chain of inequalities from (3.15):

$$\begin{aligned} \frac{1}{3} \ln \cosh x + 1 &< 2 \cosh^{1/6} x - 1 < \cosh^{1/3} x < \frac{1}{3} + \frac{2}{3} \cosh^{1/2} x \\ &< \frac{1}{2} + \frac{1}{2} \cosh^{2/3} x < \frac{28}{51} + \frac{23}{51} \cosh^{17/23} x < \frac{\sinh x}{x} < \frac{2}{3} + \frac{1}{3} \cosh x. \end{aligned}$$

Let $p = kq$. Then $\Omega_{p,q} = \{(p, q) : p \geq 0 \text{ or } 3q \leq p \leq 0\}$ is changed into

$$(3.16) \quad \Omega_{kq,q} = \{(k, q) : k, q \geq 0 \text{ or } k, q \leq 0 \text{ or } k \in [0, 3], q \leq 0\},$$

while $M(t; p, q)$ can be expressed as

$$(3.17) \quad M(t; kq, q) = \begin{cases} \left(1 - \frac{k}{3} + \frac{k}{3}t^q\right)^{1/(kq)} & \text{if } kq \neq 0, (k, q) \in \Omega_{kq,q}, \\ \exp \frac{t^q - 1}{3q} & \text{if } q \neq 0, k = 0, \\ t^{1/3} & \text{if } q = 0. \end{cases}$$

Remark 6. Similar to the monotonicity of $M(t; p, q)$, we claim that $M(t; kq, q)$ is decreasing (increasing) in q if $k > (<) 3$, and is decreasing (increasing) in k if $q > (<) 0$.

In fact, logarithmic differentiations gives

$$\begin{aligned} \frac{\partial \ln M}{\partial q} &= \frac{1}{q^2} \left(\frac{qt^q \ln t}{3 - k + kt^q} - \frac{1}{k} \ln \left(1 - \frac{k}{3} + \frac{k}{3}t^q \right) \right) := \frac{M_2(t; k, q)}{q^2}, \\ \frac{\partial M_2}{\partial q} &= \frac{t^q \ln^2 t}{(3 - k + kt^q)^2} q(3 - k), \end{aligned}$$

which means that M_2 is decreasing (increasing) in q on $(0, \infty)$ and increasing (decreasing) on $(-\infty, 0)$ if $k > (<) 3$. Hence we have $M_2(t; k, q) < (>) M_2(t; k, 0) = 0$ if $k > (<) 3$, which reveals that M is decreasing (increasing) in q for $k > (<) 3$.

Analogously, the monotonicity of $M(t; kq, q)$ with respect to k easily follows from the following relations:

$$\begin{aligned} \frac{\partial \ln M}{\partial k} &= \frac{1}{k^2} \left(\frac{k}{q} \frac{t^q - 1}{3 - k + kt^q} - \frac{1}{q} \ln \left(1 - \frac{k}{3} + \frac{k}{3}t^q \right) \right) := \frac{M_3(t; k, q)}{k^2}, \\ \frac{\partial M_3}{\partial k} &= -\frac{k}{q} \frac{(t^q - 1)^2}{(3 - k + kt^q)^2}. \end{aligned}$$

Using Corollary 1 we get

Theorem 4. *Let $x \in (0, \infty)$ and $k \in [0, 3)$. Then*

(i) when $k \in [23/17, 3)$, the inequality

$$(MI4) \quad \frac{\sinh x}{x} > \left(1 - \frac{k}{3} + \frac{k}{3} \cosh^q x\right)^{1/(kq)}$$

holds if and only if $q \leq 8/(5(3-k))$;

(ii) when $k \in [0, 1]$, the double inequality

$$(MI5) \quad \left(1 - \frac{k}{3} + \frac{k}{3} \cosh^{q_1} x\right)^{1/(kq_1)} < \frac{\sinh x}{x} < \left(1 - \frac{k}{3} + \frac{k}{3} \cosh^{q_2} x\right)^{1/(kq_2)}$$

holds if and only if $q_1 \leq 0$ and $q_2 \geq 8/(5(3-k))$.

Proof. (i) In the case of $k \in [23/17, 3)$. As shown previously, we see that the inequality (MI4) is equivalent to $D_{kq,q}(x) = Sh_{kq}(x) - (1/3)Ch_q(x) > 0$. Then, by Corollary 1, we see that (MI4) holds for $0 \leq q \leq 8/(5(3-k))$. For $q < 0$, since $M^k(t; kq, q)$ is a weighted power mean of order q of positive numbers 1 and $\cosh x$, so we have $M^k(t; kq, q) < M^k(t; k \times 0, 0)$, and then (MI4) still holds, which proves the sufficiency. The necessity can be derived from $\lim_{x \rightarrow 0^+} x^{-4} D_{kq,q}(x) \geq 0$, which by 3.6 gives

$$\lim_{x \rightarrow 0^+} \frac{D_{kq,q}(x)}{x^4} = \frac{1}{72} \left(kq - 3q + \frac{8}{5}\right) \geq 0.$$

Solving the inequality for q leads to $q \leq 8/(5(3-k))$.

(ii) In the case of $k \in (0, 1]$. The sufficiency follows from Corollary 1. It remains to prove the necessity. If the second inequality in (MI5) holds for $x \in (0, \infty)$, then by 3.6 we have

$$\lim_{x \rightarrow 0^+} \frac{D_{kq_2,q_2}(x)}{x^4} = \frac{1}{72} \left(kq_2 - 3q_2 + \frac{8}{5}\right) \leq 0,$$

which implies $q_2 \geq 8/(5(3-k))$. Lastly, we show that the condition $q_1 \leq 0$ is necessary for the first inequality in (MI5) to be true. If $q_1 > 0$, then $0 < kq_1 \leq q_1$. From (3.8) we know that $\lim_{x \rightarrow \infty} e^{-q_1 x} D_{kq_1,q_1}(x) = -2^{-q_1}/(3q_1) < 0$, which means that there is an enough large number x_N such that $D_{kq_1,q_1}(x) < 0$ for $x > x_N$, this contradict with the fact that $D_{kq_1,q_1}(x) > 0$ for $x \in (0, \infty)$.

This theorem is proved. \square

Taking $k = 1, 3/2, 2$ in Theorem 4, we get

Corollary 5. *(i) The double inequality*

$$(3.18) \quad \left(\frac{2}{3} + \frac{1}{3} \cosh^{q_1} x\right)^{1/q_1} < \frac{\sinh x}{x} < \left(\frac{2}{3} + \frac{1}{3} \cosh^{q_2} x\right)^{1/q_2}$$

holds if and only if $q_1 \leq 0$ and $q_2 \geq 4/5$.

(ii) The inequality

$$(3.19) \quad \frac{\sinh x}{x} > \left(\frac{1}{2} + \frac{1}{2} \cosh^q x\right)^{2/(3q)}$$

holds if and only if $q \leq 16/15$.

(iii) The inequality

$$(3.20) \quad \frac{\sinh x}{x} > \left(\frac{1}{3} + \frac{2}{3} \cosh^q x\right)^{1/(2q)}$$

holds if and only if $q \leq 8/5$.

Remark 7. *Part (i) in corollary above is exactly Theorem Zhu2.*

We close this section by considering the case of $p = 3q - 8/5$. In this case, $\Omega_{p,q} = \{(p, q) : p \geq 0 \text{ or } 3q \leq p \leq 0\}$ is changed into

$$(3.21) \quad \Omega_{3q-8/5,q} = \{3q - 8/5 \geq 0 \text{ or } 3q \leq 3q - 8/5 \leq 0\} = \{q \geq \frac{8}{15}\},$$

while $M(t; p, q)$ can be expressed as

$$(3.22) \quad M(t; 3q - \frac{8}{5}, q) = \begin{cases} \left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right)t^q\right)^{5/(15q-8)} & \text{if } q > \frac{8}{15}, \\ \exp \frac{5(t^{8/15}-1)}{8} & \text{if } q = \frac{8}{15}, \end{cases}$$

where $t = \cosh x \in (1, \infty)$ for $x > 0$. We assert that $M(t; 3q - 8/5, q)$ is decreasing in $q \in [8/15, \infty)$. Indeed, for $q > 8/15$, logarithmic differentiation yields

$$\begin{aligned} \frac{\partial \ln M}{\partial q} &= -3 \frac{\ln \left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right)t^q\right)}{\left(3q - \frac{8}{5}\right)^2} - \frac{\frac{8}{15q^2}(1-t^q) + t^q \left(\frac{8}{15q} - 1\right) \ln t}{\left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right)t^q\right) \left(3q - \frac{8}{5}\right)}, \\ \frac{\partial \ln M}{\partial t} &= 5q \frac{t^{q-1}}{(15q-8)t^q + 8}, \\ \frac{\partial^2 \ln M}{\partial q \partial t} &= 40t^q \frac{\ln t^q - t^q + 1}{t((15q-8)t^q + 8)^2} < 0, \end{aligned}$$

where the inequality holds due to $\ln x \leq x - 1$ for $x > 0$. Hence, $\partial(\ln M)/\partial q$ is decreasing in t , and so we have

$$\frac{\partial \ln M}{\partial q}(t; 3q - 8/5, q) < \frac{\partial \ln M}{\partial q}(1; 3q - 8/5, q) = 0,$$

which means that $q \mapsto M(t; 3q - 8/5, q)$ has decreasing property. Now we show that

$$(3.23) \quad \lim_{q \rightarrow \infty} M(t; 3q - 8/5, q) = t^{1/3}.$$

Employing L'Hospital rule yields

$$\begin{aligned} &\lim_{q \rightarrow \infty} \ln M(t; 3q - 8/5, q) \\ &= 5 \lim_{q \rightarrow \infty} \frac{\ln((15q-8)t^q + 8) - \ln(15q)}{15q-8} = \frac{1}{3} \lim_{q \rightarrow \infty} \left(\frac{t^q(15q \ln t - 8 \ln t + 15)}{15qt^q - 8t^q + 8} - \frac{1}{q} \right) \\ &= \frac{1}{3} \lim_{q \rightarrow \infty} \left(\frac{15 \ln t - 8q^{-1} \ln t + 15q^{-1}}{15 - 8q^{-1} + 8q^{-1}t^{-q}} - \frac{1}{q} \right) = \frac{1}{3} \ln t, \end{aligned}$$

that is, (3.23) is valid.

Theorem 5. *Let $x \in (0, \infty)$ and $q > 8/15$. Then the inequality*

$$(MI6) \quad \frac{\sinh x}{x} > \left(\frac{8}{15q} + \left(1 - \frac{8}{15q}\right) \cosh^q x \right)^{5/(15q-8)}$$

holds true if and only if $q \geq 34/35$. Its reverse holds if and only if $q \leq 4/5$.

Proof. The sufficiency is obviously a consequence of Corollary 2. The necessity such that (MI6) holds due to $\lim_{x \rightarrow 0^+} x^{-6} D_{3q-8/5,q}(x) \geq 0$, which together with (3.7) yields $q \geq 34/35$. It remains to treat the necessity such that the reverse of (MI6). Due to the decreasing property of $M(\cosh x; 3q - 8/5, q)$, if there is a more

large number $q^* > 4/5$ such that reverse of (MI6) holds, which is equivalent to $D_{3q^*-8/5, q^*}(x) < 0$ for $x \in (0, \infty)$, then $3q^* - 8/5 > q^* > 4/5$. From (3.8) we get $\lim_{x \rightarrow \infty} e^{-q^* x} D_{3q^*-8/5, q^*}(x) = \infty$, which implies that there is an enough large number x_N such that $D_{3q^*-8/5, q^*}(x) > 0$ for $x > x_N$. This contradict with the fact that $D_{3q^*-8/5, q^*}(x) < 0$ for $x \in (0, \infty)$, therefore, the constant $4/5$ is the best.

Thus the proof of this theorem is complete. \square

Putting $q = 34/35, 1, 16/15, 6/5, 8/5, 2, \infty$ and $4/5, 7/10, 2/3, 3/5, 8/15^+$ in Theorem 5 we have

Corollary 6. *For $x \in (0, \infty)$, the chain of inequalities hold:*

$$\begin{aligned} \cosh^{1/3} x &< \cdots < \left(\frac{11}{15} \cosh^2 x + \frac{4}{15}\right)^{5/22} < \left(\frac{2}{3} \cosh^{8/5} x + \frac{1}{3}\right)^{5/16} < \\ &\left(\frac{5}{9} \cosh^{6/5} x + \frac{4}{9}\right)^{1/2} < \left(\frac{1}{2} \cosh^{16/15} x + \frac{1}{2}\right)^{5/8} < \left(\frac{7}{15} \cosh x + \frac{8}{15}\right)^{5/7} < \\ &\left(\frac{23}{51} \cosh^{34/35} x + \frac{28}{51}\right)^{35/46} < \frac{\sinh x}{x} < \left(\frac{1}{3} \cosh^{4/5} x + \frac{2}{3}\right)^{5/4} < \left(\frac{5}{21} \cosh^{7/10} x + \frac{16}{21}\right)^2 < \\ &\left(\frac{1}{5} \cosh^{2/3} x + \frac{4}{5}\right)^{5/2} < \cdots < \exp\left(\frac{5}{8} \cosh^{8/15} x - \frac{5}{8}\right). \end{aligned}$$

4. INEQUALITIES FOR MEANS

Let G, A, Q and L stand for the geometric, arithmetic, quadratic and logarithmic means of any positive numbers a and b defined by

$$\begin{aligned} G &= G(a, b) = \sqrt{ab}, \quad A = A(a, b) = \frac{a+b}{2}, \quad Q = Q(a, b) = \sqrt{\frac{a^2+b^2}{2}}, \\ L &= L(a, b) = \frac{a-b}{\ln a - \ln b} \text{ if } a \neq b \text{ and } L = L(a, a) = a. \end{aligned}$$

The Schwab-Borchardt mean of two numbers $a \geq 0$ and $b > 0$, denoted by $SB(a, b)$, is defined as [22, Theorem 8.4], [23, 3, (2.3)]

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)} & \text{if } a < b, \\ a & \text{if } a = b, \\ \frac{\sqrt{a^2-b^2}}{\operatorname{arccosh}(a/b)} & \text{if } a > b. \end{cases}$$

The properties and certain inequalities involving Schwab-Borchardt mean can be found in [24], [25]. Very recently, Yang [26, Theorem 7.1] has defined a family of two-parameter hyperbolic sine means as follows.

Definition 1. *Let $p, q \in \mathbb{R}$ and let $Sh(p, q, t)$ be defined by*

$$(4.1) \quad Sh(p, q, t) = \begin{cases} \left(\frac{q \sinh pt}{p \sinh qt}\right)^{1/(p-q)} & \text{if } pq(p-q) \neq 0, \\ \left(\frac{\sinh pt}{pt}\right)^{1/p} & \text{if } p \neq 0, q = 0, \\ \left(\frac{\sinh qt}{qt}\right)^{1/q} & \text{if } p = 0, q \neq 0, \\ e^{t \coth pt - 1/p} & \text{if } p = q, pq \neq 0, \\ 1 & \text{if } p = q = 0. \end{cases}$$

Then for all $b \geq a > 0$, $Sh_{p,q}(b, a)$ defined by

$$(4.2) \quad Sh_{p,q}(b, a) = a \times Sh(p, q, \operatorname{arccosh}(b/a)) \text{ if } a < b \text{ and } Sh_{p,q}(a, a) = a$$

is a mean of a and b if (p, q) satisfies

$$\begin{aligned} p + q \leq 3 \text{ and } L(p, q) \leq \frac{1}{\ln 2}, & \text{ if } p, q > 0, \\ 0 \leq p + q \leq 3, & \text{ otherwise,} \end{aligned}$$

where $L(p, q)$ is the logarithmic mean of positive numbers p and q .

As a special case, for $b \geq a > 0$,

$$Sh_{1,0}(b, a) = a \frac{\sinh t}{t} \Big|_{t=\operatorname{arccosh}(b/a)}$$

is a mean of a and b . Clearly, $Sh_{1,0}(b, a) = SB(b, a)$. Thus, after replacing t by $\operatorname{arccosh}(b/a)$ and multiplying each sides of those inequalities showed in previous section by a , Theorems 2–5 still hold, for example, Theorems 2–5 can be restated as follows.

Theorem 6. Let $b \geq a > 0$ and $(p, q) \in \Omega_{p,q} = \{(p, q) : p \geq 0 \text{ or } 3q \leq p \leq 0\}$. Then

(i) when $q \in [34/35, \infty)$, the double inequality (MI2')

$$\left(\left(1 - \frac{p_1}{3q} \right) a^q + \frac{p_1}{3q} b^q \right)^{1/p_1} a^{1-q/p_1} < SB(b, a) < \left(\left(1 - \frac{p_2}{3q} \right) a^q + \frac{p_2}{3q} b^q \right)^{1/p_2} a^{1-q/p_2}$$

holds if and only if $p_1 \geq 3q - 8/5$ and $p_2 \leq q$;

(ii) when $q \in [4/5, 34/35)$, the double inequality (MI2') holds for $p_1 \geq 23q/17$ and if and only if $p_2 \leq q$;

(iii) when $q \in (0, 4/5)$, the double inequality (MI2') holds for $p_1 \geq 23q/17$ and if and only if $p_2 \leq 3q - 8/5$;

(iv) when $q \in (-\infty, 0]$, the double inequality (MI2') holds if and only if $p_1 \geq q$ and $p_2 \leq 3q - 8/5$.

Theorem 7. Let $b \geq a > 0$ and $(p, q) \in \Omega_{p,q} = \{(p, q) : p \geq 0 \text{ or } 3q \leq p \leq 0\}$. Then

(i) when $p \in [46/35, \infty)$, the double inequality (MI3')

$$\left(\left(1 - \frac{p}{3q_1} \right) a^{q_1} + \frac{p}{3q_1} b^{q_1} \right)^{1/p} a^{1-q_1/p} < SB(b, a) < \left(\left(1 - \frac{p}{3q_2} \right) a^{q_2} + \frac{p}{3q_2} b^{q_2} \right)^{1/p} a^{1-q_2/p}$$

holds if and only if $q_1 \leq p/3 + 8/15$ and $q_2 \geq p$;

(ii) when $p \in [4/5, 46/35)$, the double inequality (MI3') holds for $q_1 \leq 17p/23$ and if and only if $q_2 \geq p$;

(iii) when $p \in (0, 4/5)$, the double inequality (MI3') holds for $q_1 \leq 17p/23$ and if and only if $q_2 \geq p/3 + 8/15$;

(iv) when $p \in (-\infty, 0]$, the double inequality (MI3') holds if and only if $q_1 \leq p$ and $q_2 \geq p/3 + 8/15$.

Theorem 8. Let $b \geq a > 0$ and $k \in [0, 3)$. Then

(i) when $k \in [23/17, 3)$, the inequality

$$(MI4') \quad SB(b, a) > \left(\left(1 - \frac{k}{3} \right) a^q + \frac{k}{3} b^q \right)^{1/(kq)} a^{1-1/k}$$

holds if and only if $q \leq 8/(5(3-k))$;

(ii) when $k \in [0, 1]$, the double inequality
 (MI5')
 $\left(\left(1 - \frac{k}{3}\right)a^{q_1} + \frac{k}{3}b^{q_1}\right)^{1/(kq_1)} a^{1-1/k} < SB(b, a) < \left(\left(1 - \frac{k}{3}\right)a^{q_2} + \frac{k}{3}b^{q_2}\right)^{1/(kq_2)} a^{1-1/k}$
 holds if and only if $q_1 \leq 0$ and $q_2 \geq 8/(5(3-k))$.

Theorem 9. Let $b \geq a > 0$ and $q > 8/15$. Then the inequality

$$(MI6') \quad SB(b, a) > \left(\frac{8}{15q}a^q + \left(1 - \frac{8}{15q}\right)b^q\right)^{5/(15q-8)} a^{-q/(15q-8)}$$

holds true if and only if $q \geq 34/35$. Its reverse holds if and only if $q \leq 4/5$.

Further, let $m = m(a, b)$ and $M = M(a, b)$ be two means of a and b with $m(a, b) < M(a, b)$ for all $a, b > 0$. Clearly, making a change of variables $a \rightarrow m(a, b)$ and $b \rightarrow M(a, b)$, $Sh_{p,q}(M, m)$ is still a mean of a and b which lie in m and M . Particularly, taking $(M, m) = (A, G)$, (Q, A) , (Q, G) , respectively, we can obtain new symmetric means as follows:

$$\begin{aligned} Sh_{1,0}(A, G) &= SB(A, G) = \frac{a-b}{\ln a - \ln b} = L(a, b), \\ Sh_{1,0}(Q, A) &= SB(Q, A) = \frac{a-b}{2 \operatorname{arcsinh} \frac{a-b}{a+b}} = NS(a, b), \\ Sh_{1,0}(Q, G) &= SB(Q, G) = \frac{a-b}{\sqrt{2} \operatorname{arcsinh} \frac{a-b}{\sqrt{2ab}}} = V(a, b), \end{aligned}$$

where $NS(a, b)$ is Neuman-Sándor mean first given by [24], $V(a, b)$ is a new mean first appeared in [26].

Thus, after replacing $(b, a, SB(b, a))$ with (A, G, L) , (Q, A, NS) , (Q, G, V) , Theorems 6–8 are still true.

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